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Fixed points of asymptotic contractions

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Abstract

Let $\phi:\mathbb{R}^+ \to \mathbb{R}^+$ be a contractive gauge function in the sense that ϕ is continuous, $\phi(s) < s$ for $s > 0$, and if $f: M \to M$ satisfies $d(f(x), f(y)) \leqslant \phi(d(x, y))$ for all x, y in a complete metric space (M, d), then f always has a unique fixed point. It is proved that if $T: M \to M$ satisfies

$$d\big(T^n(x), T^n(y)\big) \leqslant \phi_n\big(d(x, y)\big), \quad x, y \in M,$$

where each ϕ_n is continuous and $\phi_n \to \phi$ uniformly on the range of d, then T has a unique fixed point, and moreover all of the Picard iterates of T converge to this fixed point.

Keywords: Contraction mappings; Asymptotic contractions; Fixed points; Approximate fixed points

1. Introduction

The usefulness of Banach's contraction mapping principle lies in the fact that the underlying space is quite general (complete metric) while the conclusion is strong, including even error estimates. On the other hand, there have been numerous extensions of a milder form of Banach's principle which conclude only that the fixed point is unique and Picard iterates of the mapping always converge to this fixed point. Probably the first to receive significant attention is the following result of Rakotch [17].

Theorem 1.1. *Let M be a complete metric space and suppose $f: M \to M$ satisfies*

$$d\big(f(x), f(y)\big) \leqslant \alpha\big(d(x, y)\big)d(x, y) \quad \text{for each } x, y \in M,$$

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where $\alpha : [0, \infty) \to [0, 1)$ *is monotonically decreasing. Then* f *has a unique fixed point,* $\bar{x}$*, and* $\{f^n(x)\}$ *converges to* $\bar{x}$ *for each* $x \in M$.

Subsequently Boyd and Wong [2] obtained a more general result. In their theorem it is assumed that $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right (that is, $r_j \downarrow r \geqslant 0 \Rightarrow \limsup_{n\to\infty} \psi(r_j) \leqslant \psi(r)$).

Theorem 1.2 [2]. *Let* M *be a complete metric space and suppose* $f : M \to M$ *satisfies*

$$d\big(f(x), f(y)\big) \leqslant \psi\big(d(x, y)\big) \quad \text{for each } x, y \in M,$$

where $\psi : [0, \infty) \to [0, \infty)$ *is upper semicontinuous from the right and satisfies* $0 \leqslant \psi(t) < t$ *for* $t > 0$*. Then* f *has a unique fixed point* $\bar{x}$*, and* $\{f^n(x)\}$ *converges to* $\bar{x}$ *for each* $x \in M$.

Our objective in this note is to introduce an 'asymptotic' version of the Boyd–Wong result. For further historical comments, see, e.g., [10].

2. Asymptotic contractions

Asymptotic fixed point theory involves assumptions about the iterates of the mapping in question. It has a long history in nonlinear functional analysis (e.g., see [5]), and in fact the concept of 'asymptotic contractions' is suggested by one of the earliest versions of Banach's principle attributed to Caccioppoli [6]. Caccioppoli's result asserts that if M is a complete metric space then the Picard iterates of a mapping $T : M \to M$ converge to the unique fixed point of T provided for each $n \geqslant 1$, there exists a constant c_n such that

$$d\big(T^n(x), T^n(y)\big) \leqslant c_n d(x, y), \quad x, y \in M,$$

with $\sum_{n=1}^{\infty} c_n < \infty$.

We now introduce a wider class of mappings that we call 'asymptotic contractions.'

Let Φ denote the class of all mappings $\phi : [0, \infty) \to [0, \infty)$ satisfying:

(i) ϕ is continuous;
(ii) $\phi(s) < s$ for all $s > 0$.

Observe that if N is any complete metric space and $f : N \to N$ is any mapping satisfying

$$d\big(f(x), f(y)\big) \leqslant \phi\big(d(x, y)\big), \quad x, y \in N,$$

for $\phi \in \Phi$, then by Theorem 1.2 f has a unique fixed point.

Definition 2.1. Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be an asymptotic contraction if

$$d\big(T^n(x), T^n(y)\big) \leqslant \phi_n\big(d(x, y)\big) \quad \text{for all } x, y \in M, \tag{2.1}$$

where $\phi_n : [0, \infty) \to [0, \infty)$ and $\phi_n \to \phi \in \Phi$ uniformly on the range of d.

The following theorem shows that asymptotic contractions typically have unique fixed points. The proof we give is nonconstructive, although there is nothing in the formulation of the theorem to suggest that such a proof is necessary.

Theorem 2.1. *Suppose (M, d) is a complete metric space and suppose $T : M \to M$ is an asymptotic contraction for which the mappings ϕ_n in (2.1) are also continuous. Assume also that some orbit of T is bounded. Then T has a unique fixed point $z \in M$, and moreover the Picard sequence $(T^n(x))_{n=1}^{\infty}$ converges to z for each $x \in M$.*

Proof. There are three preliminary steps.

Step 1. First, isometrically embed M as a closed subset of a Banach space X and identify M with its image in X. (The fact that this can be done is a classical result. In fact X can be taken to be the space of all real-valued bounded continuous functions on X. For an explicit proof see, e.g., [15, p. 234].)

Step 2. Next, let $\tilde{X}$ denote a Banach space ultrapower of X over some nontrivial ultrafilter $\mathcal{U}$ (see, e.g., [8,9]) and let $\tilde{M}$ denote the image of M in X; that is, set

$$\tilde{M} = \bigl\{\tilde{x} = \bigl[(x_n)\bigr] \in \tilde{X}\colon\ x_n \in M \text{ for each } n\bigr\}.$$

Let $\tilde{d}$ denote the metric on $\tilde{M}$ inherited from the ultrapower norm $\|\cdot\|_{\mathcal{U}}$ in $\tilde{X}$. Then $(\tilde{M}, \tilde{d})$ is a complete metric space since it is a closed subset of the Banach space $\tilde{X}$. In particular, if $\tilde{x} = [(x_n)]$, $\tilde{y} = [(y_n)] \in \tilde{M}$, then (x_n) and (y_n) are bounded sequences, so

$$\lim_{\mathcal{U}} d(x_n, y_n) = \tilde{d}(\tilde{x}, \tilde{y})$$

always exists.

Step 3. Finally, define $\tilde{T}, \hat{T} : \tilde{M} \to \tilde{M}$ as follows. For $\tilde{x} = [(x_n)] \in \tilde{M}$, set

$$\tilde{T}(\tilde{x}) = \bigl[\bigl(T(x_n)\bigr)\bigr]$$

and

$$\hat{T}(\tilde{x}) = \bigl[\bigl(T^n(x_n)\bigr)\bigr].$$

The fact that ϕ_1 is continuous assures that $\tilde{T}$ is well-defined, and the fact that the orbits of T are bounded assures that $\hat{T}$ is well-defined.

We are now in a position to take advantage of the fact that $\hat{T}$ and $\tilde{T} \circ \hat{T}$ are commuting contractions in $\tilde{M}$. Since $\phi_n \to \phi$ uniformly we have

$$\begin{aligned}\tilde{d}\bigl(\hat{T}(\tilde{x}), \hat{T}(\tilde{y})\bigr) &= \bigl\|\hat{T}(\tilde{x}) - \hat{T}(\tilde{y})\bigr\|_{\mathcal{U}} = \lim_{\mathcal{U}}\bigl\|T^n(x_n) - T^n(y_n)\bigr\| \\ &= \lim_{\mathcal{U}} d\bigl(T^n(x_n), T^n(y_n)\bigr) \leqslant \lim_{\mathcal{U}} \phi_n\bigl(d(x_n, y_n)\bigr) \\ &= \phi\Bigl(\lim_{\mathcal{U}} d(x_n, y_n)\Bigr) = \phi\bigl(\tilde{d}(\tilde{x}, \tilde{y})\bigr).\end{aligned}$$

Since $\phi \in \Phi$, $\hat{T}$ has a unique fixed point, say $\tilde{z} \in \tilde{M}$.

On the other hand,

$$\begin{aligned}
&\tilde{d}\big(\tilde{T}\circ\hat{T}(\tilde{x}),\tilde{T}\circ\hat{T}(\tilde{y})\big)=\big\|\tilde{T}\circ\hat{T}(\tilde{x})-\tilde{T}\circ\hat{T}(\tilde{y})\big\|_{\mathcal{U}}\\
&\quad=\lim_{\mathcal{U}}\big\|T^{n+1}(x_n)-T^{n+1}(y_n)\big\|=\lim_{\mathcal{U}}d\big(T^{n+1}(x_n),T^{n+1}(y_n)\big)\\
&\quad\leqslant\lim_{\mathcal{U}}\phi_{n+1}\big(d(x_n,y_n)\big)=\phi\Big(\lim_{\mathcal{U}}d(x_n,y_n)\Big)=\phi\big(\tilde{d}(\tilde{x},\tilde{y})\big).
\end{aligned}$$

Therefore $\tilde{T}\circ\hat{T}$ also has a unique fixed point and, since $\hat{T}$ and $\tilde{T}\circ\hat{T}$ commute, $\hat{T}\circ\tilde{T}(\tilde{z})=\tilde{T}\circ\hat{T}(\tilde{z})=\tilde{z}=\hat{T}(\tilde{z})$. Thus $\tilde{T}(\tilde{z})=\tilde{z}$, and from this we conclude

$$\lim_{\mathcal{U}}d\big(z_n,T(z_n)\big)=0.$$

It is now possible to extract from the sequence (z_n) a sequence (x_n) such that $\lim_n d(x_n,T(x_n))=0$.

Now suppose (y_n) is any sequence in M for which $\lim_n d(y_n,T(y_n))=0$. Then $\tilde{y}=[(y_n)]$ is also a fixed point of $\tilde{T}$. Thus, if $k\in\mathbb{N}$,

$$\begin{aligned}
\tilde{d}(\tilde{z},\tilde{y})&=\tilde{d}\big(\tilde{T}^k(\tilde{z}),\tilde{T}^k(\tilde{y})\big)=\lim_{\mathcal{U}}d\big(T^k(z_n),T^k(y_n)\big)\\
&\leqslant\lim_{\mathcal{U}}\phi_k\big(d(z_n,y_n)\big)=\phi_k\big(\tilde{d}(\tilde{z},\tilde{y})\big).
\end{aligned}$$

(The preceding step uses continuity of the mappings ϕ_k.) Letting $k\to\infty$ gives

$$\tilde{d}(\tilde{z},\tilde{y})\leqslant\phi\big(\tilde{d}(\tilde{z},\tilde{y})\big),$$

and since ϕ satisfies (ii) this in turn implies $\tilde{d}(\tilde{z},\tilde{y})=0$. This means

$$\lim_{\mathcal{U}}d(z_n,y_n)=0$$

for *any* approximate fixed point sequence (y_n) of T. Now suppose

$$\lim_n d\big(x_n,T(x_n)\big)=0\quad\text{and}\quad\lim_n d\big(y_n,T(y_n)\big)=0,$$

but $\lim_n d(x_n,y_n)\neq 0$. Then by passing to subsequences if necessary we may assume $\lim_n d(x_n,y_n)=\varepsilon>0$. This implies

$$\varepsilon=\lim_{\mathcal{U}}d(x_n,y_n)\leqslant\lim_{\mathcal{U}}d(x_n,z_n)+\lim_{\mathcal{U}}d(y_n,z_n)=0$$

which is a contradiction.

So $\lim_n d(x_n,y_n)=0$ for any pair (x_n), (y_n) of approximate fixed point sequences of T.

Now let $F_n=\{x\in M\colon\ d(x,T(x))\leqslant 1/n\}$. The foregoing implies $F_n\neq 0$ for each n, and since T is continuous each set F_n is closed. Suppose $\lim_n \operatorname{diam}(F_n)=\rho>0$. Then it is possible to choose points $x_n,y_n\in F_n$ such that $d(x_n,y_n)\geqslant\rho/2$. Since (x_n) and (y_n) are approximate fixed point sequences for T this contradicts $\lim_n d(x_n,y_n)=0$. Therefore $\lim_n\operatorname{diam}(F_n)=0$, and since M is complete Cantor's intersection theorem implies $\bigcap F_n$ is a singleton which, necessarily, is the unique fixed point of T.

Finally, suppose $\bigcap F_n=\{z\}$, let $x\in M$, and let $i\in\mathbb{N}$. Then

$$\begin{aligned}
&\limsup_n d\big(T^n(x),T^{n+1}(x)\big)=\limsup_n d\big(T^{n+i}(x),T^{n+i+1}(x)\big)\\
&\quad\leqslant\lim_n\phi_n\big(d\big(T^i(x),T^{i+1}(x)\big)\big)=\phi\big(d\big(T^i(x),T^{i+1}(x)\big)\big),
\end{aligned}$$

and, letting $i \to \infty$,

$$\lim_n d\bigl(T^n(x), T^{n+1}(x)\bigr) \leqslant \phi\bigl(\lim_n d\bigl(T^n(x), T^{n+1}(x)\bigr)\bigr)$$

from which $\lim_n d(T^n(x), T^{n+1}(x)) = 0$. Thus given any $n \in \mathbb{N}$ the sequence $(T^n(x))_{n=1}^{\infty}$ is eventually in F_n, and since the diameters of the sets F_n tend to 0 as $n \to \infty$, $\lim_n T^n(x) = z$. $\square$

Remark. In the proof of Theorem 2.1 it is not essential to embed M in a Banach space. An alternate approach would be to define $\tilde{M}$ directly as a metric space ultrapower of M and to note that $\tilde{M}$ is also complete. In fact this is the case regardless of whether M is complete (see, e.g., [3, p. 79]).

3. Historical comments

Boyd and Wong also show in [2] that if the space M is metrically convex, then the upper semicontinuity assumption on ψ can be dropped. Matkowski has extended this fact even further in [14] by showing that it suffices to assume that ψ is continuous at 0 and that there exists a sequence $t_n \downarrow 0$ for which $\psi(t_n) < t_n$.

In [12] Meir and Keeler extended the Boyd–Wong result [2] to mappings satisfying the following more general condition:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad \varepsilon \leqslant d(x, y) < \varepsilon + \delta \quad \Rightarrow \quad d\bigl(f(x), f(y)\bigr) < \varepsilon. \tag{3.1}$$

Quite recently, Lim [11] introduced the following definition.

Definition 3.1. A function $\lambda : [0, \infty) \to [0, \infty)$ is called an L-function if $\lambda(0) = 0$, $\lambda(x) > 0$ if $s > 0$, and for every $s > 0$, there exists $u > s$ such that

$$\lambda(f) \leqslant s \quad \text{for } f \in [s, u].$$

Lim's purpose in introducing the above definition was to more accurately compare the Meir–Keeler condition to that of Boyd–Wong. In particular Lim showed that a function $f : M \to M$ satisfies the Meir–Keeler condition (3.1) if and only if there exists an L-function λ such that

$$d\bigl(f(x), f(y)\bigr) \leqslant \lambda\bigl(d(x, y)\bigr) \quad x, y \in M.$$

For other extensions of Banach's theorem see, for example, Browder [4] and Matkowski [13,14]. Also see [10] for a brief survey. It is not yet clear whether these weaker versions of the contractive condition have asymptotic formulations.

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